Introduction to Simon-Smith min-max theory

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Generalized families of surfaces and saturated sets

A family $\{\Sigma_t\}_{t\in[0,1]}$, $\Sigma_t \subset M$, is a **generalized family of surfaces** (GFS) if:

- Σ_t is a surface except for a finite set $t \in \mathcal{T} \subset [0,1]$.
- There exists a finite set $\mathcal{P} \subset M$ such that $\Sigma_t \setminus \mathcal{P}$ is a surface for all $t \in \mathcal{T}$.
- $t \mapsto \mathcal{H}^2(\Sigma_t)$ is continuous.
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Let $\psi(t,x) : [0,1] \times M \to M$ be an isotopy. If $\{\Sigma_t\}$ is a GFS, then $\{\psi(t,\Sigma_t)\}$ is also a GFS.

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A collection of GFS's Λ is a **saturated set** if it is closed under the previous operation.

Inf max and minimizing sequences

Given $\{\Sigma_t\} \in \Lambda$, we define:

$$\mathcal{F}(\{\Sigma_t\}) := \max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t)$$

$$m_0(\Lambda) := \inf_{\Lambda} \mathcal{F} = \inf_{\Sigma_t\in\Lambda} \max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t)$$

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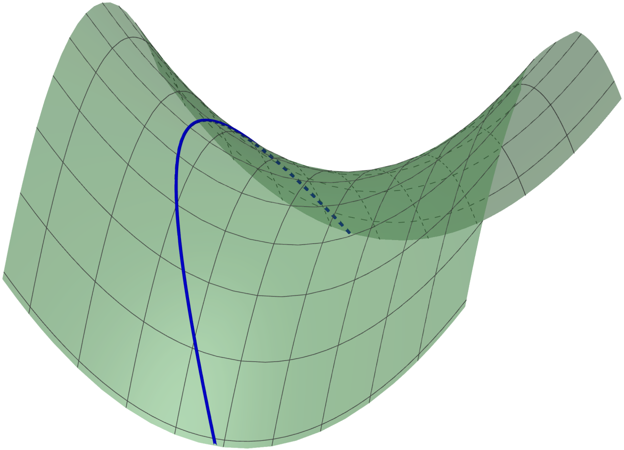
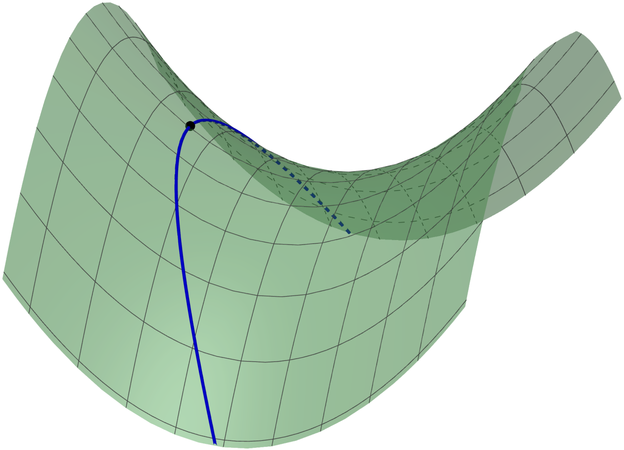
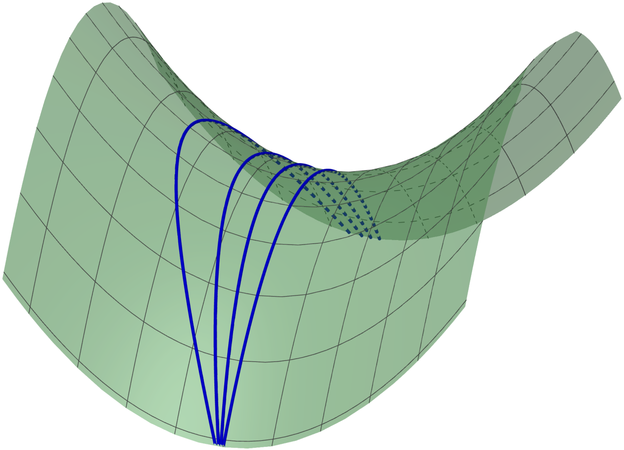
A sequence $\{\Sigma_t\}^n$ is **minimizing** if $\mathcal{F}(\{\Sigma_t\}^n) \to m_0(\Lambda)$.

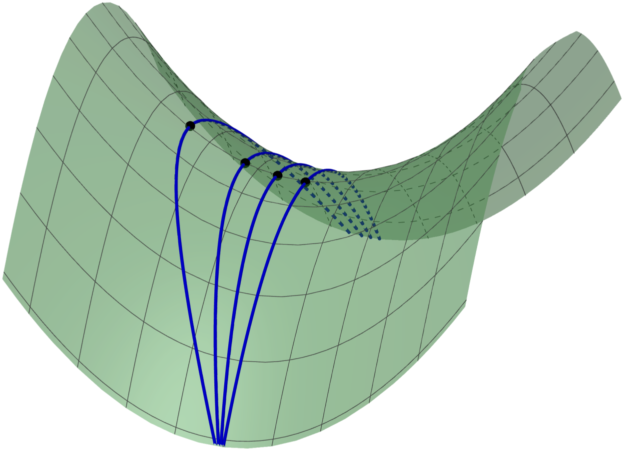
A sequence of slices $\{\Sigma^n_{t_n}\}$ is a **min-max sequence** if $\mathcal{H}^2(\Sigma^n_{t_n}) \to m_0(\Lambda)$.

To obtain minimal surfaces, we need to find a Λ such that $m_0(\Lambda) > 0$.









Constructing an appropriate saturated set

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Let $U_t = f^{-1}([0,t))$, $V_t = \psi(t, U_t)$. The volume $\text{Vol}(V_t)$ is continuous, and moreover $\text{Vol}(V_0) = 0$, $\text{Vol}(V_1) = \text{Vol}(M)$.

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In particular, there exists V_s whose volume is $\mathrm{Vol}(M)/2$. By the isoperimetric inequality and by definition of $\mathcal{F}(\{\Gamma_t\})$,

$$0 < c(M) \leq \mathcal{H}^2(\Gamma_s) \leq \mathcal{F}(\{\Gamma_t\}),$$

and so $m_0(\Lambda) \geq c(M) > 0$.

Main Theorem

Theorem [Simon-Smith]

Let M be a closed Riemannian 3-manifold. Given any saturated set Λ such that $m_0(\Lambda) > 0$, there exists a min-max sequence converging to a **embedded minimal surface** with area $m_0(\Lambda)$.

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We will find an appropriate min-max sequence converging to a **minimal surface**.

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Regular surfaces as linear operators

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$$G(U) := \bigcup_{x \in U} G(T_x M).$$

Any surface $\Sigma \subset U$ with finite area induces a (non negative) bounded linear operator on $C_c(G(U))$:

$$\varphi(x, \pi) \in C_c(G(U)) \longmapsto \int_\Sigma \varphi(x, T_x\Sigma) d\mathcal{H}^2$$

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Varifolds: a weak notion of surfaces

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Moreover, there exists a unique measure $\|V\|$, called the **mass measure**, defined on U, and such that given $\varphi \in C_c(U)$,

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If Σ is a surface and V_Σ is its associated varifold, then $\|V_\Sigma\|(U)$ is the area of Σ in U.

The space of varifolds and its compactness properties

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is **metrizable** and **compact**. In particular, any sequence of varifolds with uniformly bounded mass has a **convergent subsequence**.

First variation and stationary varifolds

Let $f : U \to U'$ be a diffeomorphism and $V \in \mathcal{V}(U)$. Then, f **induces** a varifold $f_{\#}V \in \mathcal{V}(U')$.

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We define the **first variation** of a varifold V w.r.t. χ as

$$[\delta V](\chi) = \frac{d}{dt}(\|\psi_\chi(t,\cdot)_\# V\|)\Big|_{t=0}.$$

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We say that V is **stationary** if $[\delta V](\chi) = 0$ for every field χ.

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Finding stationary varifolds

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Theorem (Pull-tight process)

There exists a **minimizing sequence** $\{\Gamma_t\}^n$ such that **any** min-max sequence $\{\Gamma^n_{t_n}\}$ converges to a stationary varifold.

Idea of the Theorem: For each varifold we will define an isotopy $\psi_V(t,x)$ such that:

- If V is stationary, then $\psi_V(t,\cdot)$ is the identity map.
- Otherwise, $V' := (\psi_V(1,\cdot))_\# V$ has strictly less mass than V.
- The difference between $\|V'\|(M)$ and $\|V\|(M)$ depends uniformly on the distance between V and the set of stationary varifolds.

Pull-tight process

Let X be the set of varifolds with mass less or equal than $4m_0$, which is **compact** and **metrizable**.

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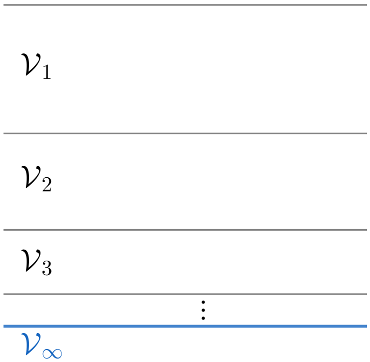
The sets $\mathcal{V}_k$ are also **compact**.



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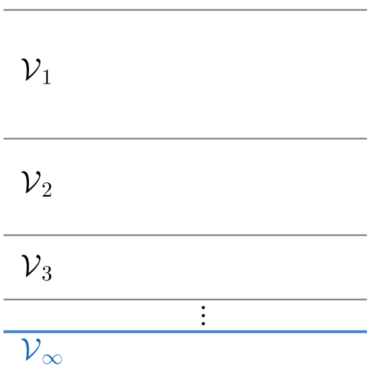
There exists $c = c(k) > 0$ such that, for each $V \in \mathcal{V}_k$ there exists χ_V with $\|\chi_V\| \leq 1$ and

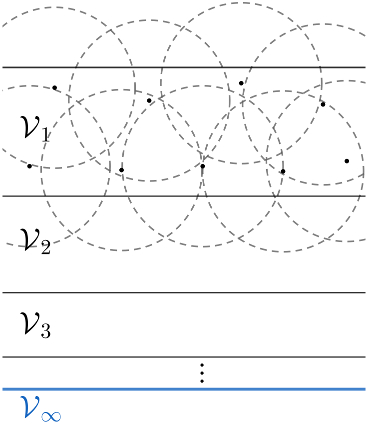
$$[\delta V](\chi_V) \leq -c(k).$$



Pull-tight process

By **compactness**, we can take a finite set of varifolds $V_i^k \subset \mathcal{V}_k$, fields χ_i^k and balls $B(V_i^k, r_i^k)$ s.t.:

- If $V \in B(V_i^k, r_i^k)$, then $[\delta V](\chi_i) \leq -c(k)/2$.



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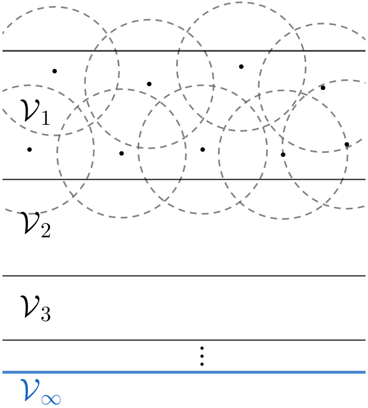

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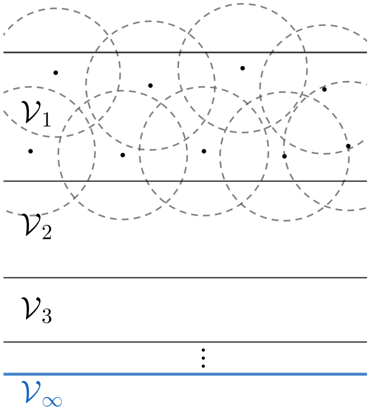
Let $\varphi_i^k \in C_c(B_{r_i^k}(V_i^k))$ be a partition of the unit, and define $H : X \to C^\infty(M, TM)$ as

$$H(V) := \sum_{i,k} \varphi_i^k(V)\chi_i^k.$$

Notice that H is **continuous**.

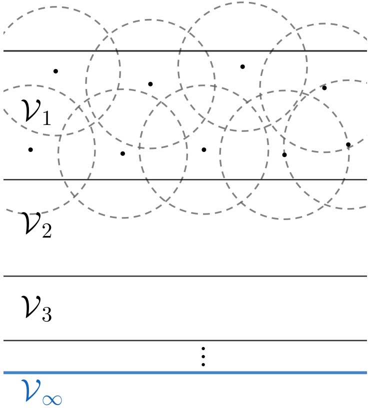


Pull-tight process

There exist constants $C = C(k)$ and a time $T = T(k)$ such that

$$\|V(T)\|(M) \leq \|V(0)\|(M) - C(k)$$

for every $V \in \mathcal{V}_k$, where $V(T)$ is the *evolution* of $V = V(0)$ under the field $H(V)$.



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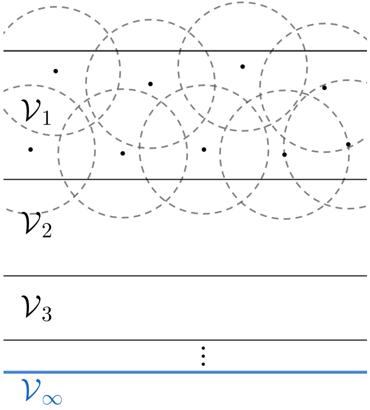
Now, let $\{\Sigma_t\}^n$ be a minimizing sequence. Then, we can define a *tighter* minimizing sequence:

Let $\Gamma_t^n := \Sigma_t^n(T)$. Then, the sequence $\{\Gamma_t\}^n$ is minimizing and each min-max sequence converges to a **stationary varifold**.



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Let $\varepsilon > 0$, $U \subset M$ and $\Sigma \subset U$. We will say that the surface Σ is ε-almost minimizing if there **does not** exist any isotopy ψ supported in U such that:

- $\mathcal{H}^2(\psi(t, \Sigma)) \leq \mathcal{H}^2(\Sigma) + \frac{\varepsilon}{8}$ for all $t \in [0, 1]$.
- $\mathcal{H}^2(\psi(1, \Sigma)) \leq \mathcal{H}^2(\Sigma) - \varepsilon$.

Almost minimizing surfaces

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A sequence $\{\Sigma^n\}$ is almost minimizing if each Σ^n is ε_n-almost minimizing, and $\varepsilon_n \to 0$.

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A sequence $\{\Sigma^n\}$ is almost minimizing if each Σ^n is ε_n-almost minimizing, and $\varepsilon_n \to 0$.

Let $\mathcal{AN}(x,r)$ be the set of annuli centered in $x \in M$ with *outer radius* less than r.

A sequence $\{\Sigma^n\}$ is almost minimizing in small annuli if there exists $r : M \to (0,\infty)$ such that $\{\Sigma^n\}$ is almost minimizing in every $An \in \mathcal{AN}(x, r(x))$.

Replacements

Let V be a stationary varifold and $U \subset M$. We say that V' is a **replacement** of V in U if:

- V' is stationary and $\|V\| = \|V'\|$.
- $V = V'$ on $M \setminus U$.
- $\Sigma := V'|_U$ is a embedded stable minimal surface with $\overline{\Sigma} \setminus \Sigma \in \partial U$.

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We say that V has the **good replacement property** if:

- V has a replacement V' in any $An \in \mathcal{AN}(x, r(x))$.
- V' has a second replacement V'' in any $An \in \mathcal{AN}(y, r(y))$.
- V'' has a third replacement V''' in any $An \in \mathcal{AN}(z, r(z))$.

Regularity results

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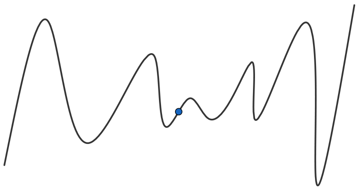
Theorem 3 (a.m. min-max sequence has GRP)

The varifold V of the previous Proposition has the good replacement property. In particular, V is an **embedded, minimal surface** with area $m_0(\Lambda)$.

Talk structure

Replacements of V

If V is stationary and admits replacements in small annuli, then it is **integer rectifiable**.



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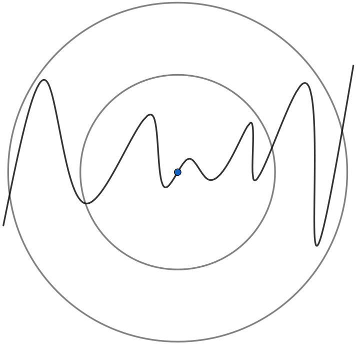
We fix some $x \in M$, $\rho > 0$ and consider the annulus $An_1 := An(x, \rho, 2\rho)$.



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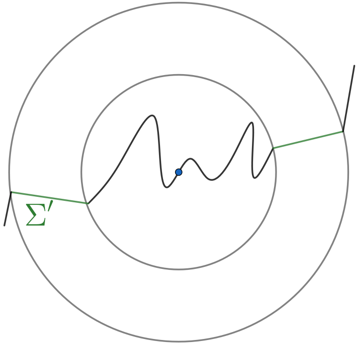

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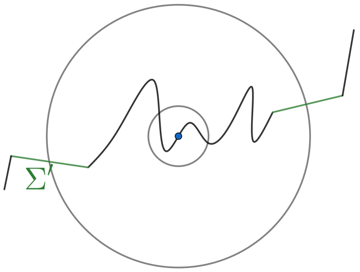

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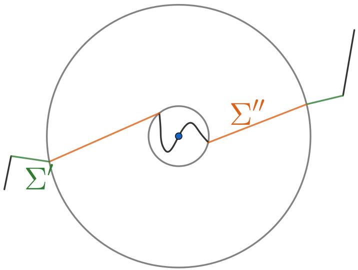

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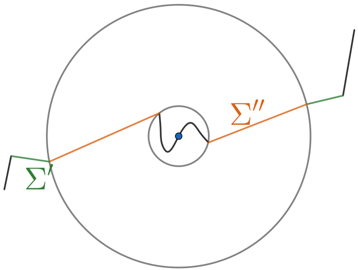

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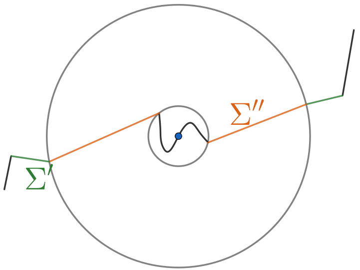
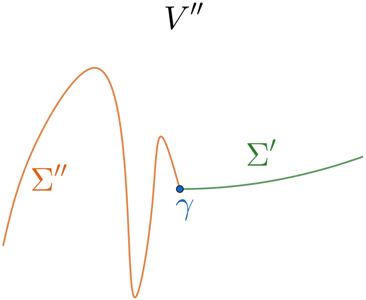
We have two surfaces Σ' and Σ''. **Do they coincide?**



Gluing two minimal surfaces

Let $\gamma := \Sigma' \cap \partial B_t$, and take t so that Σ' meets ∂B_t transversally. One can check that V'' has a tangent plane along γ, which must coincide with that of Σ'.



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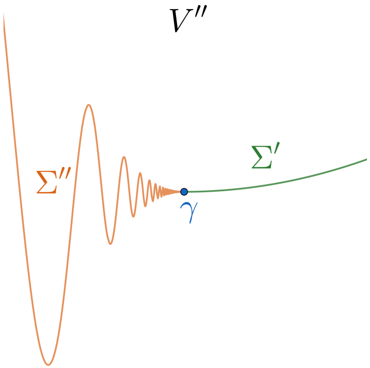

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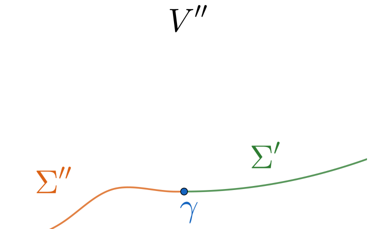

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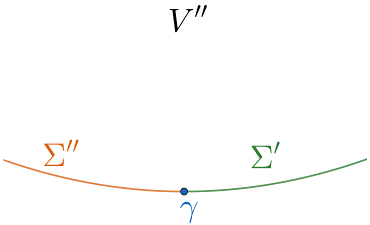
By PDE theory, since Σ', Σ'' and their Gauss maps coincide along γ, $\Sigma' \equiv \Sigma''$.



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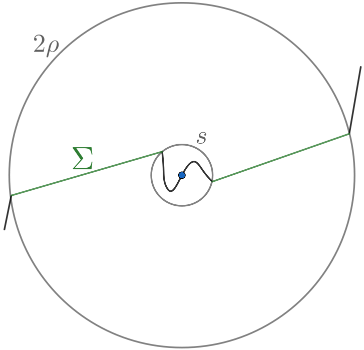

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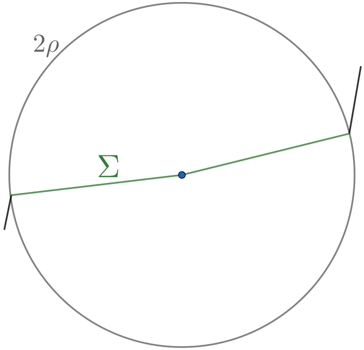

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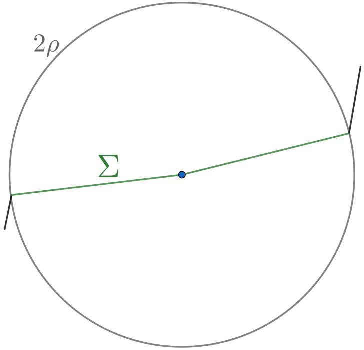

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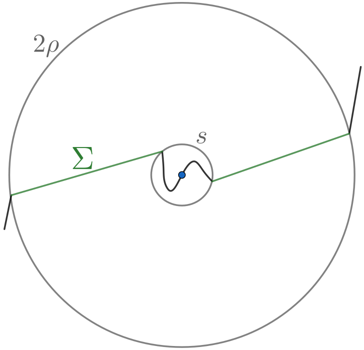

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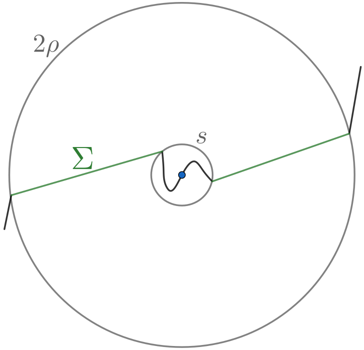

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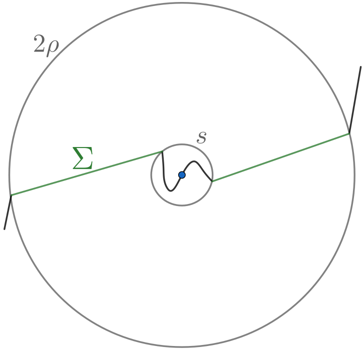
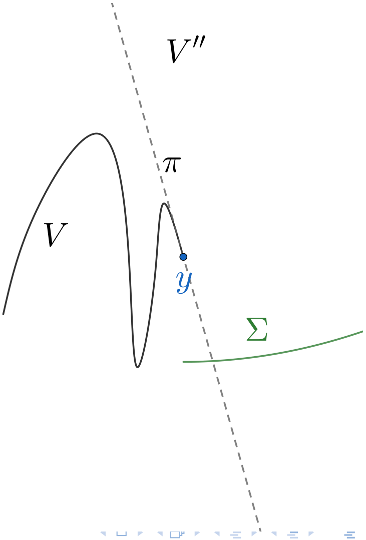
Let $y \in \mathrm{Supp}(\|V\|) \cap B^*(x,\rho)$ s.t. V meets $\partial B(x,s)$ **transversally**, where $s := d(x,y)$. In particular, there exists a plane $\pi \in T_y M$ tangent to V.



$V = \Sigma$ on a punctured ball

Since $V = V''$ in $B(x, s)$, π is tangent to V''.



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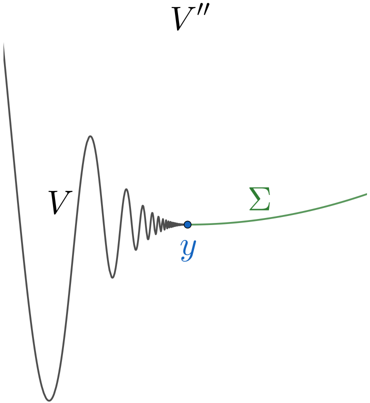

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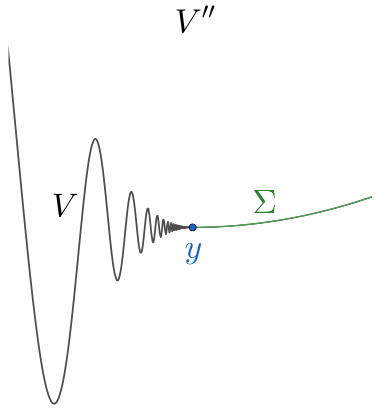
The set of points $y \in \mathrm{Supp}(\|V\|) \cap B^*(x, \rho)$ transversal to $\partial B(x, s)$ is **dense**.



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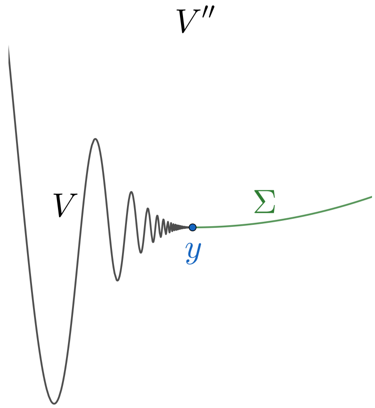

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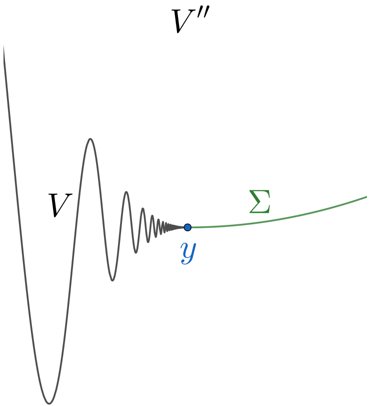

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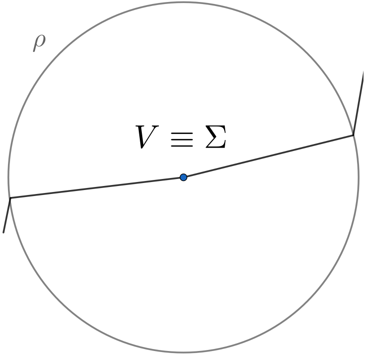

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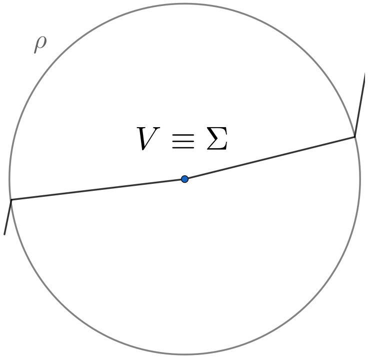
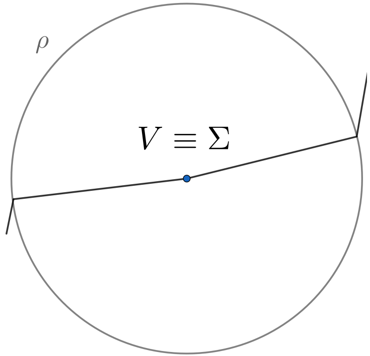
As a result, $V|_{B^*(x,\rho)} = \Sigma$. Now, can we extend Σ smoothly to x?



We know that the tangent varifold to V at x is a **plane** π with multiplicity M.



Smooth extension of Σ at x

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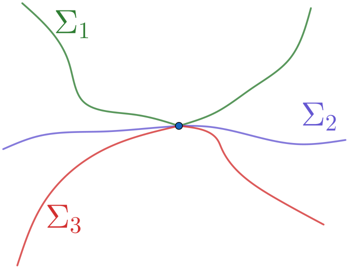
Stability shows that near x, there are minimal Lipschitz graphs Σ_i and constants m_i, $1 \leq i \leq N$ with $\sum m_i = M$ and

$$\Sigma = \sum m_i \Sigma_i$$

near x. Moreover, Σ_i are ordered by height.



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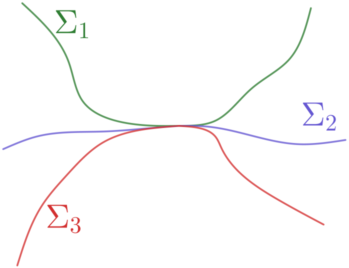

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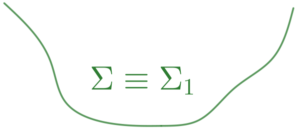

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Talk structure

Sketch of the proof

We need to prove that there exists a min-max sequence $\{\Sigma^n\}$ s.t.:

1. It converges to a stationary varifold,
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We deduce that there is $r = r(x)$ such that every annulus An with outer radius less than $r(x)$ does not contain any P_i nor P_i^n for large n.

Finding an almost minimizing min-max sequence

Let $\mathcal{CO}$ be the set of pairs (U_1, U_2) such that

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In any of the cases, we obtain a min max subsequence $\{\Sigma^{L(j)}\}$ which is a.m. in small annuli.

Proof of the Proposition

Let $L \in \mathbb{N}$, and

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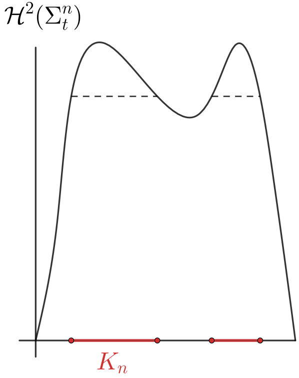

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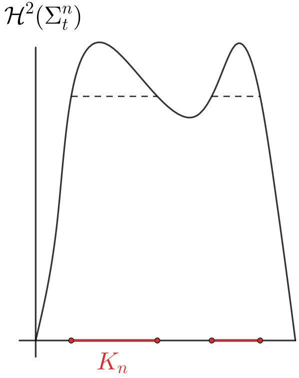

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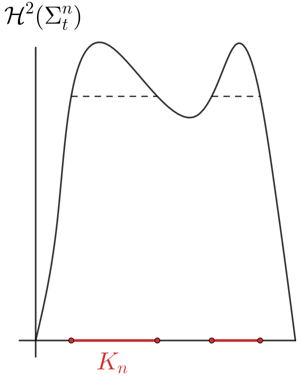

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Let $L \in \mathbb{N}$, and

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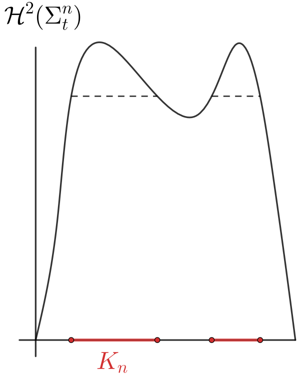
We can find at least two isotopies ψ_U and ψ_V which decrease the area of Σ_t^n with small increase in the process.



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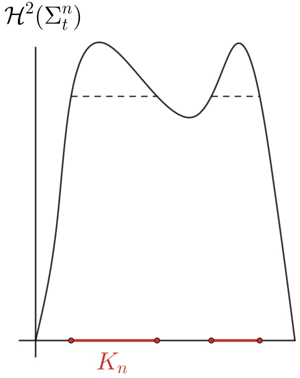

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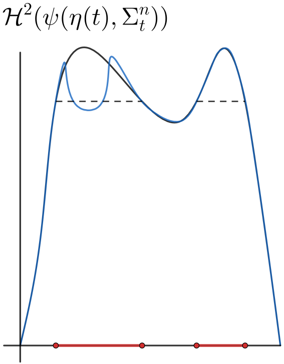
Idea: apply one of the isotopies ψ_U, ψ_V along I.



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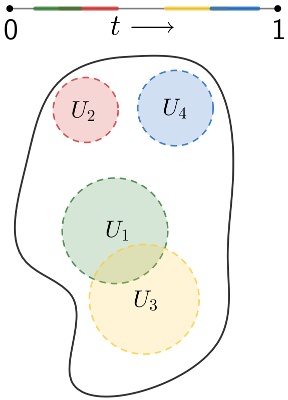
Now, we take a finite cover $\{I_j\}_{1\leq j\leq N}$ of K_n by some of these intervals, each associated with an isotopy $\psi_j(\eta_j, \cdot)$ supported in U_j.



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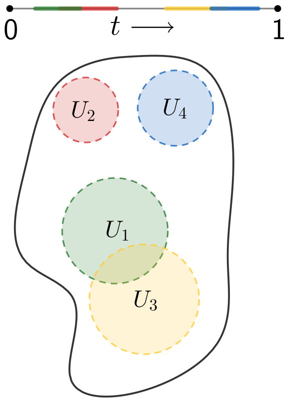
- $I_j \cap I_k = \varnothing$ unless $|j-k| = 1$.
- The sets U_j, U_{j+1} are **disjoint** if $I_j \cap I_{j+1}$ overlap.
- For all $t \in K_n$, one of the $\eta_j(t)$ is 1.



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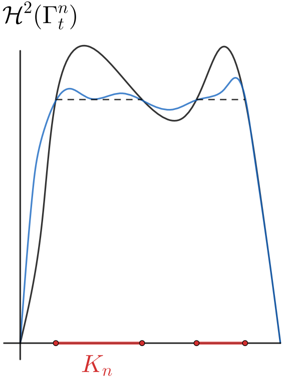

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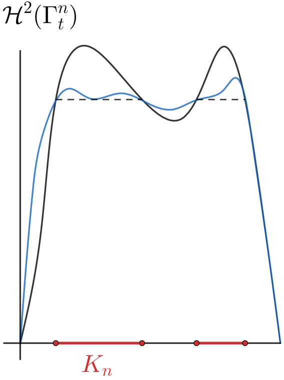
After applying the isotopies $\psi(\eta_j(t),\cdot)$ to $\{\Sigma_t^n\}$, we obtain a new family Γ_t^n satisfying:

$$\mathcal{F}(\{\Gamma_t^n\}) \leq \mathcal{F}(\{\Sigma_t^n\}) - 1/2L.$$



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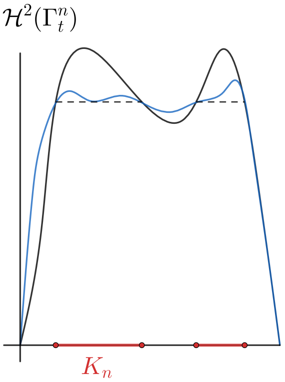
In particular, $\lim_n \mathcal{F}(\{\Gamma^n_t\}) \leq m_0(\Lambda) - 1/2L$!!



Talk structure

Preliminary definitions and results

Let $\mathcal{I}$ be a set of smooth isotopies on M and Σ be a surface. We say that $\{\psi^k(1,\Sigma)\} \subset \mathcal{I}$ is **minimizing** for $(\Sigma, \mathcal{I})$ if

$$\lim_k \mathcal{H}^2(\psi^k(1,\Sigma)) = \inf_{\psi \in \mathcal{I}} \mathcal{H}^2(\psi(1,\Sigma)).$$

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Given $U \subset M$ and an **embedded** surface Σ, we define:

- $\mathfrak{Is}(U)$: the set of all smooth isotopies supported in U.
- $\mathfrak{Is}_j(U) := \{\psi \in \mathfrak{Is}(U) \ : \ \mathcal{H}^2(\psi(t,\Sigma)) \leq \mathcal{H}^2(\Sigma) + \frac{1}{8j}\}$.

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Theorem (Meeks-Simon-Yau)

Let $\{\Sigma^k\} \subset \mathfrak{Is}(U)$ be **minimizing** and converging to a varifold V. Then, $V|_U$ is an **stable, embedded, minimal surface**.

Proof of Theorem 3

Let $\{\Sigma^j\}$ be the a.m. min-max sequence obtained in Theorem 2, which converges to a stationary varifold V.

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Any limit V^* of a subsequence of $\{V^j\}$ is a replacement for V in An.

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We assume Lemma B:

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- V^* is stationary: Otherwise, the V^j's could not be the limit of a minimization problem.

Proof of Lemma B

Fix j, and let $V' := V^j$ be the limit of $\Sigma^k := \Sigma^{j,k}$. We want to prove that $V'|_{An}$ is a stable, embedded, minimal surface. The proof is based on Lemma A:

Let $x \in An$ and k large enough. There exists $\varepsilon > 0$ s.t. any isotopy $\varphi \in \mathfrak{Is}(B_\varepsilon(x))$ can be achieved via an isotopy $\Phi \in \mathfrak{Is}_j(B_{2\varepsilon}(x))$.

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Proof of Lemma B: we show that V' is a minimal surface using **replacements**.

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We can now define a diagonal sequence $\{\Sigma^{k,l(k)}\}^k$ converging to W. Applying Lemma A, we obtain a second replacement.

Proof of Lemma B

Fix j, and let $V' := V^j$ be the limit of $\Sigma^k := \Sigma^{j,k}$. We want to prove that $V'|_{An}$ is a stable, embedded, minimal surface. The proof is based on Lemma A:

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Proof of Lemma B: we show that V' is a minimal surface using **replacements**.

Let $An^* \in \mathcal{AN}(x,\varepsilon)$ and $\{\Sigma^{k,l}\}^l$ be minimizing for $\mathfrak{Is}(An^*)$. By [MSY], the limit W^k is **minimal** in An^*, and so is $W = \lim_k W^k$. By Lemma A, $\Sigma^{k,l}$ belong to $\mathfrak{Is}_j(B_{2\varepsilon})$. We deduce that W is a replacement for V'.

We can now define a diagonal sequence $\{\Sigma^{k,l(k)}\}^k$ converging to W. Applying Lemma A, we obtain a second replacement. Iterating this process, we deduce the good replacement property.

Proof of Lemma A

Let $x \in An$ and k large enough. There exists $\varepsilon > 0$ s.t. any isotopy $\varphi \in \mathfrak{Is}(B_\varepsilon(x))$ can be achieved via an isotopy $\Phi \in \mathfrak{Is}_j(B_{2\varepsilon}(x))$.

Proof of Lemma A

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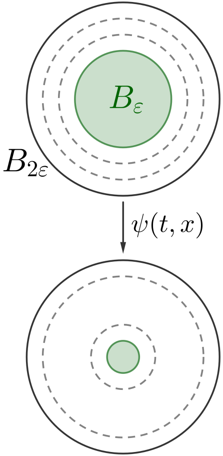
Proof: Let $\psi(t, x)$ be an isotopy which *squeezes* the ball $B_\varepsilon(x)$ in $B_{2\varepsilon}(x)$ with scale factor $(1 - t)$.



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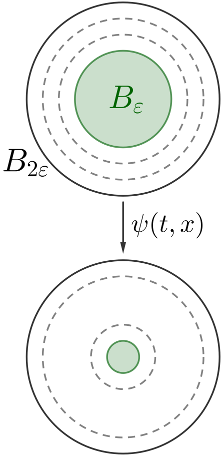
For large enough k, we can find a certain $\psi(t,x)$ satisfying

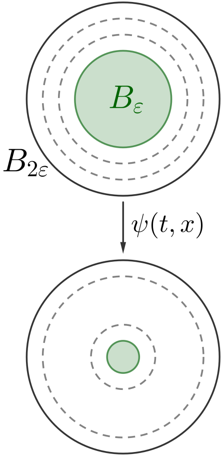
$$\mathcal{H}^2(\psi(t,\Sigma^k)) \le \mathcal{H}^2(\Sigma^k) + C\varepsilon^2.$$



Proof of Lemma A

Now, given $\varphi \in \mathfrak{Is}(B_\varepsilon(x))$, let Φ be the isotopy which applies this process:

- First, it *squeezes* B_ε via $\psi(t, x)$ up to a certain factor $(1 - t_0)$.
- Then, it applies the isotopy φ on the *squeezed ball* $B_{(1-t_0)\varepsilon}(x)$.
- Finally, it *enlarges* $B_\varepsilon(x)$ by applying $\psi(t, x)$ reversely.



Proof of Lemma A

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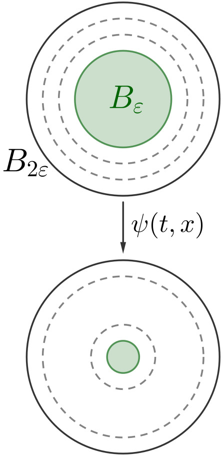
$\varphi(1, x) \equiv \Phi(1, x)$. Moreover, if t_0 is sufficiently close to 1, $\Phi(t, x) \in \mathfrak{Is}_j(B_{2\varepsilon}(x))$.



Thank you for your attention!

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